B.Math.Hons. Ist year
Ist Midsemestral exam 2005

Algebra I : Instructor - B.Sury

Answer eight questions including the first. Be brief.

1. State (**don't prove**) whether the following are true or false.
(i) The elements xy and yx are conjugate for all $x, y \in G$.
(ii) If elements a, b of finite order in a group commute, then $O(ab) = O(a)O(b)$.
(iii) It is possible for a group to be isomorphic to a proper subgroup.
(iv) A finite group in which each element has order a power of a fixed prime p, must itself have order a power of p.
(v) A finite group in which every element is a product of elements of order 2, must have order a power of 2.

2. Prove that, if $G =< g >$ is a cyclic group of order n, then every subgroup must be of the form $< g^d >$ for some divisor d of n. Use this to deduce that $n = \sum_{d|n} \phi(d)$.

3. (i) In any group G, prove that $O(xy) = O(yx)$ for all $x, y \in G$.
(ii) If G is a group in which $x^2 = e$ for all x, prove that G must be abelian.

4. If H is a subgroup of a group G such that every left coset of H is a right coset of H, then show that H must be normal.

5. (i) If $\sigma_1, \cdots, \sigma_r \in S_n$ are disjoint cycles of lengths $n_1, \cdots, n_r$, then prove that the order of $\sigma_1 \cdots \sigma_r$ is the LCM (least common multiple) of $n_1, \cdots, n_r$.
(ii) If $\sigma, \tau \in S_n$ are r-cycles for some r, prove that they are conjugate in S_n.

6. (i) If a finite group G acts on a finite set S, show that each orbit has cardinality a divisor of $O(G)$.
(ii) Let G be a finite group of order p^n for some prime p acting on a finite set S whose cardinality is not a multiple of p. Using (i) or otherwise, show that G fixes some point of S.

7. Let G be a group such that the quotient group $G/Z(G)$ is cyclic. Prove that G must be abelian.

8. If H, K are subgroups of finite indices of a group G, then prove that $H \cap K$ must have finite index $\leq [G : H][G : K]$.

9. Find the center of the group $SL_n(\mathbf{C}) = \{g \in GL_n(\mathbf{C}) : det\ g = 1\}$.

10. Consider the action of $\mathbf{Z}_n^*$ on $\mathbf{Z}_n$ by : $(a, b) \mapsto ab$. Show that the number of orbits is the number $d(n)$ of divisors of n.

11. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix with integer entries. Consider the homomorphism θ from $\mathbf{Z} \times \mathbf{Z}$ to itself given by $(x, y) \mapsto (ax + by, cx + dy)$. Prove that, if the subgroup Im (θ) has finite index in $\mathbf{Z} \times \mathbf{Z}$, then $ad - bc \neq 0$.